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On the Stability of Bilinear Stochastic Systems<sup>1</sup>

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**Abstract:** We study the stability with probability one of the stochastic bilinear system  $dX = AX ds + BX dw$ , where  $A$  and  $B$  are fixed matrices and  $w$  is a Brownian motion. Bounds for the Lyapunov numbers associated with this equation are given.

Bilinear noise models are, after linear ones, the second simplest case of stochastic systems; they may arise in many problems in which linear noise models are inappropriate (many examples are given in [6]).

The aim of this paper is to give a condition for the stability with probability one of the  $d$ -dimensional Ito equation which describes the behavior of such a system

$$\begin{aligned} dY_s &= AY_s ds + BY_s dw_s \\ Y_0 &= y \end{aligned} \tag{1}$$

where  $A$  and  $B$  are two given  $d \times d$  matrices and  $w$  is a scalar Brownian motion (see also the more general equation (12) below). I.e., we want to find whether or not  $Y_s$  tends a.e. to zero as  $s$  tends to infinity. Note that in the one dimensional case, we have an explicit solution

$$Y_s = Y_0 \exp\{t(A - B^2/2) + Bw_t\},$$

and the stability with probability one is guaranteed iff  $2A - B^2 < 0$ . Note also that

$$E[(Y_s)^2] = (Y_0)^2 \exp 2t(A + B^2/2)$$

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and the  $L^2$  stability is guaranteed iff  $2A+B^2 < 0$ . This is a much stronger condition whose generalisation in  $d$  dimensions is easy, thanks to the following equations:

$$dY_s \otimes Y_s = (I \otimes A + A \otimes I + B \otimes B) Y_s \otimes Y_s ds + (I \otimes B + B \otimes I) Y_s \otimes Y_s dw_s$$

$$dE[Y_s \otimes Y_s] = (I \otimes A + A \otimes I + B \otimes B) E[Y_s \otimes Y_s] ds$$

(for the definition and the basic properties of  $\otimes$ , see [1]). The  $L^2$  stability is then governed by  $\lambda_{\max}(I \otimes A + A \otimes I + B \otimes B)$  (in this paper,  $\lambda_{\max}(M)$  denotes the largest real part of eigenvalues of the matrix  $M$ , and  $\lambda_{\min}(M) = -\lambda_{\max}(-M)$  ).

We will give an upper bound for the largest Lyapunov number  $\lambda_1$  of (1) (the smallest  $\gamma$  satisfying the bound of theorem1). Note that the only existing expressions for the largest Lyapunov number of (1) or (12) with general matrices are asymptotic expansions ( $B = \varepsilon B_0$ ,  $\varepsilon$  tends to 0,  $A$  fixed) in dimension 2 ([5]). The following criterion will be proved :

Theorem1. Setting

$$\gamma = \frac{1}{2} \lambda_{\max}(I \otimes (A - B^2) + (A - B^2) \otimes I + B \otimes B),$$

then, for any value of  $Y_0$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|Y_t\|) \leq \gamma \quad \text{a.e.}$$

Considering the matrix equation

$$dP_s = AP_s ds + BP_s dw_s$$

(2)

$$P_0 = I,$$

$Y_s$  can be expressed as

$$Y_s = P_s Y_0$$

and the following basic theorem about Lyapunov numbers (the  $\lambda_i$ 's below) is true:

Theorem2([2]): *There exist  $d$  real numbers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$  such that*

$$\lim_{t \rightarrow \infty} \{P_t^T(\omega)P_t(\omega)\}^{1/2t} = O^T(\omega)\Lambda O(\omega) \text{ a.e.}$$

where  $O(\omega)$  is an orthogonal matrix and  $\Lambda$  is the diagonal matrix satisfying

$$\Lambda_{i,i} = \exp(\lambda_i).$$

Furthermore, if

$$E_i = \text{span}\{O^T(\omega)e_i, O^T(\omega)e_{i+1}, \dots, O^T(\omega)e_d\}$$

( $(e_i)_{i=1,d}$  is the canonical basis of  $\mathbb{R}^d$ ) then, a.e., we have

$$\forall u \in E_i(\omega) \setminus E_{i+1}(\omega), \quad \lim_{t \rightarrow \infty} \frac{1}{t} \text{Log}\|P_t(\omega)u\| = \lambda_i.$$

All the  $\lambda_i$  are equal if and only if there exist a matrix  $M$  such that the matrices  $M(A - \frac{1}{d} \text{trace}(A))M^{-1}$  and  $M(B - \frac{1}{d} \text{trace}(B))M^{-1}$  are skew-symmetric.

Note that the Lyapunov numbers of  $Q_s = P_s^{-T}$  are  $(-\lambda_d, \dots, -\lambda_1)$ .

To prove theorem1, instead of trying to find upper bounds on the Lyapunov numbers of  $P_s$ , we will get lower bounds for the ones of  $Q_s$ . Theorem2 shows that the smallest Lyapunov number is generally attained on a random 1-dimensional space; this fact is at the origin of all the difficulty of step1 below. The motivation of this method will appear in step2, when the (non-quadratic) term  $(u^T F u)^2$  in (3) will be bounded from below by  $u^T F^T F u$  (dealing with  $P_s$ , we would have to find an upper bound for  $(u^T B u)^2$ ).

Proof of theorem1

$c$  denotes a constant depending only on  $d$  and is allowed to change, finitely often, during the calculation.

Note that  $Q_s = P_s^{-T}$  satisfies the equation

$$dQ_s = EQ_s ds + FQ_s dw_s$$

where

$$E = -A^T + B^2T$$

$$F = -B^T.$$

By virtue of theorem2 (and the following remark), the largest Lyapunov number of  $P_s$  is a.e.

$$-\lim_{t \rightarrow \infty} \inf_x \frac{1}{t} \log(\|Q_t x\|).$$

For any  $X_0$ ,  $X_t = Q_t X_0$  satisfies the equation

$$dX_s = EX_s ds + FX_s dw_s$$

Using Ito's formula, and setting  $u_s = X_s / \|X_s\|$ , we obtain

$$dX_s^T X_s = X_s^T X_s (2u_s^T E u_s ds + 2u_s^T F u_s dw_s + u_s^T F^T F u_s ds)$$

$$d \log(\|X_s\|) = \{ u_s^T E u_s + \frac{1}{2} u_s^T F^T F u_s - (u_s^T F u_s)^2 \} ds + u_s^T F u_s dw_s.$$

This equation, due to Khas'minski ([3]), is basic for the study of Lyapunov numbers, since it may be written

$$t^{-1} \{ \log(\|X_t\|) - \log(\|X_0\|) \} = t^{-1} \int_0^t \{ u_s^T E u_s + \frac{1}{2} u_s^T F^T F u_s - (u_s^T F u_s)^2 \} ds + t^{-1} \int_0^t u_s^T F u_s dw_s$$

(3)

step1:  $E[\sup_{X_0} |\int_0^t u_s^T F u_s dw_s|] = o(t), \quad t \text{ integer } > 0.$

The sup may be taken over a dense subset of  $\mathbb{R}^d$  (because the stochastic integral is a.e. a continuous function of  $X_0$ ) so that the expectation is well defined.

This is the most difficult step of the proof. All our efforts will be focused on getting rid of the sup. We will suppose that the  $\lambda_i$  are not all equal (if they are, using theorem2, the expression above is  $w_t$  times a constant and the result is obvious).

The following lemma is needed

**Lemma** *Let  $\sigma$  be a uniform measure on the unit sphere  $S_{d-1}$  of  $\mathbb{R}^d$ ,  $x$  any vector of  $\mathbb{R}^d$ ,  $f \in L^1(\sigma)$ ,  $P$  an invertible matrix, and  $Q=P^{-T}$ , the following equalities are true:*

$$\frac{x}{\|x\|} = c_1 \int \text{sign}(\langle x, y \rangle) y \sigma(dy) \quad c_1 = (\sigma(\|y\|=1))^{-1} \quad (4)$$

$$\int f(y) \sigma(dy) = \det(P) \int f\left(\frac{Py}{\|Py\|}\right) \|Py\|^{-d} \sigma(dy) \quad (5)$$

$$\frac{x^T Q^T F Q x}{\|Qx\|^2} = c_1^2 \int \text{sign}(\langle x, y \rangle) \text{sign}(\langle x, z \rangle) (z^T P^T F P y) \|Py\|^{-d-1} \|Pz\|^{-d-1} \det(P)^2 \sigma(dy) \sigma(dz) \quad (6)$$

**Proof** of the lemma:

Observe that identity (4) has only to be proved for  $\|x\|=1$ , and that, because of the rotational invariance of  $\sigma$ , we can suppose  $x=(1,0,0,\dots,0)$ , and we get  $c_1$ .

The left-hand side of (5) is equal to

$$\begin{aligned} c \int f\left(\frac{y}{\|y\|}\right) e^{-\|y\|} dy &= c \int f\left(\frac{Px}{\|Px\|}\right) e^{-\|Px\|} \det(P) dx \\ &= c \det(P) \int f\left(\frac{Py}{\|Py\|}\right) e^{-r\|Py\|} r^{d-1} \sigma(dy) dr \\ &= c \det(P) \int f\left(\frac{Py}{\|Py\|}\right) \|Py\|^{-d} \sigma(dy) \end{aligned}$$

the constant  $c$ , independent of  $P$ , is identified by taking  $P=I$ .

In order to prove (6), use (4) and (5) to obtain

$$\begin{aligned} \frac{Qx}{\|Qx\|} &= c \int \text{sign}(\langle x, Q^T y \rangle) y \sigma(dy) \\ &= \det(P) \int \text{sign}(\langle x, y \rangle) Py \|Py\|^{-d-1} \sigma(dy) \end{aligned}$$

which implies (6).

This lemma gives us

$$u_s^T F u_s = \frac{X_0^T Q_s^T F Q_s X_0}{\|Q_s X_0\|^2} = \int \Phi(X_0, y, z) \Psi(P_s, y, z) \sigma(dy) \sigma(dz)$$

where

$$\Phi(x, y, z) = \text{sign}(\langle x, y \rangle) \text{sign}(\langle x, z \rangle)$$

$$\Psi(P, y, z) = c (z^T P^T F P y) \|P y\|^{-d-1} \|P z\|^{-d-1} \det(P)^2.$$

And

$$\int_0^t u_s F u_s^T dw_s = \int_0^t \int \Phi(X_0, y, z) \Psi(P_s, y, z) \sigma(dy) \sigma(dz) dw_s$$

$$= \int \Phi(X_0, y, z) \int_0^t \Psi(P_s, y, z) dw_s \sigma(dy) \sigma(dz)$$

$$\sup_{X_0} \left| \int_0^t u_s F u_s^T dw_s \right| \leq \int \left| \int_0^t \Psi(P_s, y, z) dw_s \right| \sigma(dy) \sigma(dz)$$

$$E \left\{ \sup_{X_0} \left| \int_0^t u_s F u_s^T dw_s \right| \right\} \leq \int E \left[ \left\{ \int_0^t \Psi^2(P_s, y, z) ds \right\}^{1/2} \right] \sigma(dy) \sigma(dz)$$

$$E \left\{ \sup_{X_0} \left| \int_0^t u_s F u_s^T dw_s \right| \right\} \leq E \left[ \int_0^t \left\{ \int_0^t \Psi^2(P_s, y, z) ds \right\}^{1/2} \sigma(dy) \sigma(dz) \right] \quad (7)$$

we will prove that for almost all  $(\omega, y, z)$ ,

$$\int_0^\infty \Psi^2(P_s, y, z) ds < \infty \quad (8)$$

and that the integrals

$$\frac{1}{t} \left( \int_0^t \Psi^2(P_s, y, z) ds \right)^{1/2} \quad (9)$$

are uniformly (on  $t$ ) integrable (over  $(\omega, y, z)$ ). And then (7), (8), and (9) will imply the bound of step1.

To prove (8), note that

$$\begin{aligned} |\Psi(P_s, y, z)| &= c (z^T P_s^T F P_s y) \|P_s y\|^{-d-1} \|P_s z\|^{-d-1} \det(P_s)^2 \\ &\leq c \|F\| \|P_s y\|^{-d} \|P_s z\|^{-d} \det(P_s)^2 \\ &\leq c \|F\| \exp\{(-2\lambda_1 d + 2 \sum \lambda_i + \varepsilon(s))s\} \end{aligned}$$

where  $\varepsilon(s) \rightarrow 0$  as  $s \rightarrow \infty$  and the  $\lambda_i$  are the Lyapounov numbers associated with  $P$ . Since we are in a case where the  $\lambda_i$  are not all equal, we get the exponential decrease of the expression above, and (8) is true.

To prove the uniform integrability of (9), we will show that

$$E \left[ \int_0^t a \left( \frac{1}{t} \left( \int_0^t \Psi^2(P_s, y, z) ds \right)^{1/2} \right) \sigma(dy) \sigma(dz) \right] < K \quad (10)$$

where  $K$  is a constant independent of  $t$ , and

$$a(x) = |x| \log^{1/2}(1+x^2).$$

Denoting

$$\psi(t, y, z) = \int_0^t |\Psi(P_s, y, z)|^2 ds,$$

equation (10) reduces to

$$E \left[ \int_0^t \frac{1}{t} \psi(t, y, z)^{1/2} \log^{1/2}(1+\psi(t, y, z)/t^2) \sigma(dy) \sigma(dz) \right] < K.$$

Step1 will be finished by proving

$$E \left[ \int \psi(t, y, z)^{1/2} \log^{1/2}(1+\psi(t, y, z)) \sigma(dy) \sigma(dz) \right] < Kt. \quad (11)$$



An elementary calculation shows that the function  $x^{1/2}\log^{1/2}(1+x)$  is concave; this implies

$$(x+y)^{1/2}\log^{1/2}(1+x+y) \leq x^{1/2}\log^{1/2}(1+x) + y^{1/2}\log^{1/2}(1+y) \quad \text{for any } x, y > 0$$

Defining  $\psi_1$  by

$$\psi_1(t, y, z) = \psi(t+1, y, z) - \psi(1, y, z) = \int_1^{t+1} |\Psi(P_s, y, z)|^2 ds$$

we obtain

$$\begin{aligned} & E \left[ \int \psi(t+1, y, z)^{1/2} \log^{1/2}(1+\psi(t+1, y, z)) \sigma(dy) \sigma(dz) \right] \\ & \leq E \left[ \int \psi(1, y, z)^{1/2} \log^{1/2}(1+\psi(1, y, z)) \sigma(dy) \sigma(dz) \right] + \\ & \quad E \left[ \int \psi_1(t, y, z)^{1/2} \log^{1/2}(1+\psi_1(t, y, z)) \sigma(dy) \sigma(dz) \right]. \end{aligned}$$

We will now get the index 1 out of the last formula. The basic tool is the following identity resulting from Markov property

$$P_{s+1} = (\theta_1 P_s) P_1$$

where  $\theta_1$  is the shift operator on the trajectories of Brownian motion. Using this identity in

$$\psi_1(t, y, z) = \int_0^t |\Psi(P_{s+1}, y, z)|^2 ds$$

and

$$\Psi(P, y, z) = c (z^T P^T F P y) \|P y\|^{-d-1} \|P z\|^{-d-1} \det(P)^2,$$

we get

$$\psi_1(t, y, z) = \int_0^t |\Psi(\theta_1 P_s, P_1 y, P_1 z)|^2 ds \det(P_1)^2.$$

The second term of the right-hand side may be rewritten as

$$E \left[ \int \psi(t, P_1 y, P_1 z)^{1/2} \log^{1/2}(1+\psi(t, P_1 y, P_1 z) \det(P_1)^4) \det(P_1)^2 \sigma(dy) \sigma(dz) \right]$$

where  $P_1$  and  $\psi(t,y,z)$  are independent(i.e. constructed from two independent Brownian motions and the expectation is taken over the product space). Using lemma1 we obtain, after some reductions ( $Q_1 = P_1^{-1}$ )

$$E[\int \psi(t,y,z)^{1/2} \log^{1/2}(1+\psi(t,y,z)\det(P_1)^4 \|Q_1 y\|^{-2d} \|Q_1 z\|^{-2d}) \sigma(dy)\sigma(dz)]$$

and with the inequalities ( $a,b>0$ )

$$\begin{aligned} \log(1+ab) &\leq \log(1+a) + \log(1+b) \\ (a+b)^{1/2} &\leq a^{1/2} + b^{1/2} \end{aligned}$$

we get the upper bound

$$\begin{aligned} E[\int a(\psi(t,y,z)^{1/2}) \sigma(dy)\sigma(dz) ] + \\ E[\int \log^{1/2}(1+\det(P_1)^4 \|Q_1 y\|^{-2d} \|Q_1 z\|^{-2d}) \sigma(dy)\sigma(dz)] \end{aligned}$$

The last term is finite since  $\log(\|Q_s y\|)$  satisfies equation (3) and  $\det(P_s)$  satisfies

$$d\log(\det(P_s)) = \{ \text{trace}(A) - 1/2 \text{ trace}(B^2) \} ds + \text{trace}(B) dw_s.$$

Finally, we have

$$E[\int a(\psi(t+1,y,z)^{1/2}) \sigma(dy)\sigma(dz) ] \leq E[\int a(\psi(t,y,z)^{1/2}) \sigma(dy)\sigma(dz) ] + K$$

where  $K$  is a constant. This ends the proof of (10), and the first step.

step2: end of the proof.

Using step1, one can find a sequence  $t_n$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \sup_{X_0} \left| \int_0^{t_n} u_s F u_s^T dw_s \right| = 0 \quad \text{a.e.}$$

and obtain, with equation (3):

$$\lim_{n \rightarrow \infty} \inf_{X_0} \frac{1}{t_n} \log(\|X_{t_n}\|) \geq \inf_{u \in S_{d-1}} u^T E u + \frac{1}{2} u^T F^T F u - (u^T F u)^2.$$

But

$$(Fu - u(u^T Fu))^T (Fu - u(u^T Fu)) \geq 0$$

implies

$$u^T F^T F u \geq (u^T F u)^2,$$

which gives the bound

$$\begin{aligned} \lim_{t \rightarrow \infty} \inf_{X_0} \frac{1}{t} \log(\|X_t\|) &\geq \inf_{u \in S_{d-1}} u^T E u - \frac{1}{2} u^T F^T F u \\ &\geq \lambda + \inf_{u \in S_{d-1}} u^T (E - \lambda I) u - \frac{1}{2} u^T F^T F u \end{aligned}$$

for any real  $\lambda$ .

Actually, we have still the choice of the basis of  $\mathbb{R}^d$  we are working in. A change of basis with a matrix  $R$  changes  $E$  into  $RER^{-1}$  and  $F$  into  $RFR^{-1}$  and then for any invertible matrix  $R$ , we have

$$\lim_{t \rightarrow \infty} \inf_{X_0} \frac{1}{t} \log(\|X_t\|) \geq \lambda + \inf_{u \in S_{d-1}} u^T R(E - \lambda I)R^{-1}u - \frac{1}{2} u^T R^{-T} F^T R^T R F R^{-1} u$$

$1_u$

$$\geq \lambda - \sup_{Rv \in S_{d-1}} v^T S(-E + \lambda I)v + \frac{1}{2} v^T F^T S F v$$

where  $S = R^T R$  ( $u = Rv$ ). One can prove ([4]) that, if

$$\lambda_{\max}(I \otimes (-E + \lambda I) + (-E + \lambda I) \otimes I + F \otimes F) < 0,$$

then the equation

$$S(-E + \lambda I) + (-E^T + \lambda I)S + F^T S F = -I$$

has a unique solution which is positive definite. Then, for any  $\lambda$  smaller than  $\lambda_{\min}(I \otimes E + E \otimes I - F \otimes F)/2$ , we have

$$\lim_{t \rightarrow \infty} \inf_{X_0} \frac{1}{t} \log(\|X_t\|) \geq \lambda$$

The smallest Lyapunov number of  $Q_s$  is then larger than  $\lambda_{\max}(I \otimes E + E \otimes I - F \otimes F)/2$ . But

$$I \otimes E + E \otimes I - F \otimes F = -\{ I \otimes (A - B^2) + (A - B^2) \otimes I + B \otimes B \}^T$$

This ends the proof of the theorem.

We have just proved the following result

Theorem 3: *The smallest Lyapunov number of equation (1) is larger than*

$$\delta = \frac{1}{2} \lambda_{\min}(I \otimes A + A \otimes I - B \otimes B).$$

If equation (1) is replaced by

$$dY_s = AY_s ds + \sum_{i=1}^n B_i Y_s dw_s^i \quad (12)$$

where  $B_1, \dots, B_n$ , are  $n$  matrices and  $w^1, \dots, w^n$  are  $n$  independent Brownian motions, the proof can still be carried out in the same way and we obtain

Theorem 1(general form): *If equation (12) is substituted to equation (1), theorem 1 and 3 are still valid with*

$$\gamma = \frac{1}{2} \lambda_{\max}(I \otimes A + A \otimes I - \sum_{i=1}^n I \otimes B_i^2 + B_i^2 \otimes I - B_i \otimes B_i)$$

$$\delta = \frac{1}{2} \lambda_{\min}(I \otimes A + A \otimes I - \sum_{i=1}^n B_i \otimes B_i)$$

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